## THE UNIVERSITY OF AKRON

## Mathematics and Computer Science

Lesson 2: Exponents \& Radicals

## Directory

- Table of Contents
- Begin Lesson 2


## ora <br> $\operatorname{Iam~WS}$

$$
\begin{gathered}
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\
a^{3} a^{4}=a^{7}(a b)^{10}=a^{10} b^{10} \\
-(a b-(3 a b-4))=2 a b-4 \\
(a b)^{3}\left(a^{-1}+b^{-1}\right)=(a b)^{2}(a+b) \\
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
2 x^{2}-3 x-2=(2 x+1)(x-2) \\
\frac{1}{2} x+13=0 \Longrightarrow x=-26 \\
G=\{(x, y) \mid y=f(x)\} \\
f(x)=m x+b
\end{gathered}
$$

Copyright © 1995-2000 D.P. Story
Last Revision Date: 2/2/2000

## Lesson 2: Exponents \& Radicals Table of Contents

2. Exponents and Radicals
2.1. Integer Exponents
2.2. The Law of Exponents

- How to Multiply or Divide Two Powers - How to Calculuate a Power of a Product or Quotient • How to Compute a Power of a Power
2.3. Radicals
- Properties of Radicals
2.4. Fractional Exponents


## 2. Exponents \& Radicals

This lesson is devoted to a review of exponents, radicals, and the infamous "Laws of Exponents." The student must have the skills to manipulate exponents without error.

### 2.1. Integer Exponents

Let $a$ be a number, and $n \in \mathbb{N}$ be a natural number. The symbol $a^{n}$ is defined as

$$
\begin{equation*}
a^{n}=\underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text { factors }} \tag{1}
\end{equation*}
$$

That is, $a^{n}$ is the product of $a$ with itself $n$ times.
Sometimes, negative exponents enter into the mix. These are defined by

$$
\begin{equation*}
a^{-n}=\frac{1}{a^{n}} \quad \text { where } n \in \mathbb{N} \text { and } a \neq 0 \tag{2}
\end{equation*}
$$

Needless to say, we define $a^{0}=1$, for all $a \neq 0$.

Thus, the symbol $a^{k}$ is defined for all integers $k \in \mathbb{Z}$ : for positive integers as in equation $(1), 2^{3}=(2)(2)(2)=8$; for negative integers as in equation (2), $2^{-3}=1 / 2^{3}=1 / 8$; and for zero, $2^{0}=1$.

Terminology. The symbol $a^{k}$ is called a power of $a$. We say that $a^{k}$ has a base of $a$ and that $k$ is the exponent of the power of $a$.

Numerical calculations offer no challenge to the student (that's you). The more interesting case is when there are symbolic quantities involved; however, there is one situation involving numerics (and symbolics) in which some students-I'm not saying you necessarily-have a weakness. Consider the following ...

Quiz. Suppose you wanted to square the number -3 , what would be the correct notational way of writing that?
(a) $-3^{2}$
(b) $(-3)^{2}$
(c) (a) and (b) are equivalent

To effectively manipulate expressions involving symbolics, we must be the masters of the Laws of Exponents - to be taken up shortly; just now, however, I want to illustrate how the definitions of $a^{k}$ are applied.

Illustration 1. Here are several important illustrations of the techniques revolving about the definitions given in equations (1) and (2).

$$
\begin{array}{lll}
\begin{array}{ll}
\text { (a) } x^{-6}=\frac{1}{x^{6}} & \text { (b) } \frac{1}{y^{12}}=y^{-12}
\end{array} & \text { (c) }(a b)^{-1}=\frac{1}{a b} \\
\text { (d) } \frac{x^{4}}{y^{-4}}=x^{4} y^{4} & \text { (e) } \frac{x^{4} y^{-2}}{z^{-5}}=\frac{x^{4} z^{5}}{y^{2}} & \text { (f) } \frac{w^{9}}{s^{-5} t^{-3}}=w^{9} s^{5} t^{3}
\end{array}
$$

Illustration Notes: Look at (a) first from left-to-right. Basically, this says that if we have a negative exponent in the numerator, we can shove the expression into the denominator by changing the sign of the exponent. Reading (a) from right-to-left, we see that if we have an expression in the denominator, we can lift it to the numerator by changing the sign of the exponent. Similar comments can be made in equation (b).

- Illustration (c) suggests that if an expression is grouped, then when we move it to the denominator, we move the whole group.
- Examples (d), (e) and (f) demonstrate how equations (1) and (2) are used in practice. Note that these expressions only involve multiplication and/or division-most important!
- Key Point. When moving expressions from the numerator to the denominator (or from denominator to numerator), simply change the sign of the exponent.
- Important. The last point is only valid for expressions involving multiplication and/or division. For example, the following shall be referred to as an algebraic blunder!

$$
\text { NOT TRUE! } \Longrightarrow \frac{1}{x^{-2}+y^{-2}}=x^{2}+y^{2} \Longleftarrow \text { NOT TRUE! }
$$

Exercise 2.1. Remove all negative exponents for each of the expressions below.
(a) $e^{-4}$
(b) $\frac{1}{(x+y)^{-6}}$
(c) $(s t)^{-(k+2)}$
(d) $\frac{w^{6}}{x^{4} y^{-7}}$
(e) $\frac{a^{7} b^{-9} c^{-1}}{d^{-4} e^{6}}$
(f) $\frac{(x+y)^{6}}{x^{-1} y^{-2}}$

### 2.2. The Law of Exponents

When the exponent in (1) is a natural number, the following equations should be manifestly obvious:

The Law of Exponents-Junior Grade:
Let $a$ and $b$ be numbers, and $n$ and $m$ be integers, then

1. $a^{n} a^{m}=a^{n+m}$
2. $(a b)^{n}=a^{n} b^{n}$ and $\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$
3. $\left(a^{n}\right)^{m}=a^{n m}$

Let's take each of these in turn, discuss them and illustrate them.

## - How to Multiply or Divide Two Powers

When multiplying two powers together with different bases, there is no simplification possible. For example, $a^{4} b^{5}$ cannot be simplified.

When multiplying two powers together having the same base, the governing law is

$$
\begin{equation*}
a^{n} a^{m}=a^{n+m} \tag{3}
\end{equation*}
$$

The validity of this identity simply follows from the definition. These calculations assume $n$ and $m$ are natural numbers.

$$
\begin{aligned}
a^{n} a^{m} & =\underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text { factors }} \underbrace{a \cdot a \cdot a \cdot a \cdots a}_{m \text { factors }} \\
& =\underbrace{a \cdot a \cdot a \cdot a \cdots a \cdot a \cdot a \cdot a \cdot a \cdots a}_{n+m \text { factors }} \\
& =a^{n+m} .
\end{aligned}
$$

Stare at this "proof." If you can keep a visualization of this demonstration in your head, then you cannot misuse this particular law.

Key Point. When multiplying powers with the same base, just add the exponents. The rule $a^{n} a^{m}=a^{n+m}$ is valid for both positive and negative integers.

Illustration 2. Here is a list of quick illustrations of this first law.
(a) $4^{9} 4^{3}=4^{12}$
(b) $x^{3} x^{5}=x^{8}$
(c) $5 y^{-3} y^{7}=5 y^{-3+7}=5 y^{4}$
(d) $\frac{1}{x^{3} x^{-8}}=\frac{1}{x^{-5}}=x^{5}$
(e) $(s+1)^{4}(s+1)^{-3}=(s+1)$ (f) $x^{4} y^{3} x^{-1} y^{-4}=x^{3} y^{-1}=\frac{x^{3}}{y}$

The illustrations above all had numerical values for the exponents. One can apply this first law even when the exponent is symbolic.

Illustration 3. Study these examples.
(a) $4^{n} 4^{n+1}=4^{2 n+1}$
(b) $(x+1)^{3 n-1}(x+1)^{4 n+3}=(x+1)^{7 n+2}$
(c) $w^{5 k} s^{4 m} w^{3 k+1} s^{5}=w^{8 k+1} s^{4 m+5}$

In each case, the base of the exponentials is the same, so we just add the exponents.

Exercise 2.2. Combine exponents as appropriate using law \#1. First solve, write down your answers using good notation, then look at the answers. (A score of $100 \%$ is expected.)
(a) $x^{4} x^{4} y^{2}$
(b) $(s+1)^{12}(s+1)(s+1)^{3}$
(c) $23 y^{3} w^{4} w^{8} y^{2}$
(d) $4^{2} 4^{9}$
(e) $x^{4 m+3} x^{5 m-1}$
(f) $y^{3 j} y^{12 j+m}$
(g) $a^{6} a^{-3}$
(h) $(a b)^{7}(a b)^{4}(a b)^{-2}$
(i) $\frac{x^{5} y^{9} y^{-2}}{x^{2} y^{3}}$

Now consider $a^{n} / a^{m}$ in the case of $m>n$. This means that there are more factors of $a$ in the denominator than the numerator. The same cancellation goes on, but there factors of $a$ left over in the denominator when all the cancellation is over. Thus,

$$
\frac{a^{n}}{a^{m}}=\frac{1}{a^{m-n}} \quad m>n
$$

Section 2: Exponents \& Radicals
Illustration 4. Simplify each.
(a) $\frac{x^{5}}{x^{12}}=\frac{1}{x^{12-5}}=\frac{1}{x^{7}}$
(b) $\frac{43 t(s+2)^{4}}{t^{4}(s+2)^{15}}=\frac{43}{t^{3}(s+2)^{11}}$
(c) $\frac{(s t)^{7}}{12 s^{12} t^{4}}=\frac{s^{7} t^{7}}{12 s^{12} t^{4}}=\frac{1}{12} \frac{s^{7}}{s^{12}} \frac{t^{7}}{t^{4}}=\frac{1}{12} \frac{1}{s^{5}} \frac{t^{3}}{1}=\frac{t^{3}}{12 s^{5}}$

The last example can be simplified in fewer steps; these additional steps were include for clarity.

The formula $a^{n} a^{m}=a^{n+m}$ can be read from right-to-left as well. Below are some examples that would suggest when it would be advantageous to use the first law in this way.

## Illustration 5.

(a) Simplifying exponentials having a negative sign in the base.

1. $(-x)^{3}=((-1) x)^{3}=(-1)^{3} x^{3}=-x^{3}$
2. $(-x)^{4}=((-1) x)^{4}=(-1)^{4} x^{4}=x^{4}$
3. $(-2 w)^{3}=(-2)^{3} w^{3}=-8 w^{3}$

Section 2: Exponents \& Radicals
4. $(-3 z)^{-2}=(-3)^{-2} z^{-2}=\frac{1}{(-3)^{2}} \frac{1}{z^{2}}=\frac{1}{9} \frac{1}{z^{2}}=\frac{1}{9 z^{2}}$
(b) Here are some abstract versions of the same type. Let $n \in \mathbb{Z}$ be an integer.

1. $(-x)^{2 n}=((-1) x)^{2 n}=(-1)^{2 n} x^{2 n}=x^{2 n}$. For any integer $n, 2 n$ is an even integer. The number -1 raised to an even power is 1 !
2. $(-t)^{2 n+1}=((-1) t)^{2 n+1}=(-1)^{2 n+1} t^{2 n+1}=-t^{2 n+1}$. For any integer $n, 2 n+1$ is an odd integer. The number -1 raised to an odd power is -1 .

This kind of simplification is seen all the time in mathematics. Learn how to master it.

Exercise 2.3. Simplify each of the following.
(a) $(-(w+1))^{13}$
(b) $(-3 w)^{4}$
(c) $(-2 s)^{-3}$
(d) $(-u)^{2 n}$
(e) $(-p)^{2 n-1}$

Study these illustrations and exercises. See if you can get a "feel" for the pattern.

Always strive to develop good algebraic techniques. Examples of good techniques are these tutorials. Don't be lazy. Think of me sitting here typing out all these examples and exercises. If I can sit here typing out examples of good algebraic notation and techniques, can't you take a little time out and do the same?

The Cancellation Law. The process of combining ratios of exponentials having the same base can be greatly accelerated by the following observations:

1. If $n>m$ (that is, the exponent in the numerator is greater than the exponent in the denominator), then

$$
\frac{a^{n}}{a^{m}}=a^{n-m}
$$

2. If $n<m$, (that is, the exponent is the denominator is greater than the exponent in the numerator), then

$$
\frac{a^{n}}{a^{m}}=\frac{1}{a^{m-n}} .
$$

When Law \#2 is viewed this way, it is referred to as a Cancellation Law.

Here is a 'proof by example.'

$$
\frac{x^{5}}{x^{2}}=x^{5} x^{-2}=x^{5-2}=x^{3}
$$

Or, if you have a quick mind, you can write

$$
\frac{x^{5}}{x^{2}}=x^{5-2}=x^{3}, \text { or just } \frac{x^{5}}{x^{2}}=x^{3}
$$

Here, the numerator has the greatest exponent, so we subtract the exponent in the denominator from the exponent in the numerator, and leave the result in the numerator.

Now consider this calculation:

$$
\frac{x^{3}}{x^{12}}=x^{3} x^{-12}=x^{3-12}=x^{-9}=\frac{1}{x^{9}}
$$

This process can be accelerated to

$$
\frac{x^{3}}{x^{12}}=\frac{1}{x^{12-3}}=\frac{1}{x^{9}}, \text { or to just } \frac{x^{3}}{x^{12}}=\frac{1}{x^{9}}
$$

The exponent in the denominator is greater than the one in the numerator, we subtract the exponent in the numerator from the exponent in the denominator, and leave the result in the denominator.

Can you latch onto the visual feel of this operation?
Illustration 6. Consider the following. Simplify each.
(a) $\frac{x^{7}}{x^{4}}=x^{7-4}=x^{3}$.
(b) $\frac{12(w+3)^{23}}{(w+3)^{3}}=12(w+3)^{20}$.

Section 2: Exponents \& Radicals
(c) $\frac{x^{5} y^{7}}{y^{3}}=x^{5} y^{4}$.
(d) $\frac{x^{5}}{x^{12}}=\frac{1}{x^{12-5}}=\frac{1}{x^{7}}$
(e) $\frac{43 t(s+2)^{4}}{t^{4}(s+2)^{15}}=\frac{43}{t^{3}(s+2)^{11}}$
(f) $\frac{s^{7} t^{7}}{12 s^{12} t^{4}}=\frac{1}{12} \frac{s^{7}}{s^{12}} \frac{t^{7}}{t^{4}}=\frac{1}{12} \frac{1}{s^{5}} \frac{t^{3}}{1}=\frac{t^{3}}{12 s^{5}}$

Illustration Notes: The last example is meant to delineate all the formal steps that go into the reasoning process. Of course, it can be simplified in fewer steps; these additional steps were included for clarity.

- Look over each of these illustrations. They all involve multiplication and division. This is the only time you can apply the laws of the exponents!
- Some expressions above, you might say, do involve additions! What do you have to say about that! I say, that these additions are enclosed with parentheses. In this case, the calculation is treated as
a single algebraic object or unit. That object is related to the rest of the expression by multiplications and/or divisions.
- As you can see, this is a process of cancellation.

Exercise 2.4. Try a few simplifications on your own. Use good notation, be neat and be correct, and ... be quick and efficient.

$$
\text { (a) } \frac{4 x^{5} y^{6}}{x^{3} y^{9} z^{2}} \text { (b) } \frac{w(u+v)^{9}}{w^{5}(u+v)^{3}} \text { (c) } \frac{x^{5} y^{5} z^{5}}{5 x^{4} y^{5} z^{12}}
$$

- How to Calculate a Power of a Product or Quotient The second Law of Exponents states that

$$
(a b)^{n}=a^{n} b^{n} .
$$

This tells us how to compute a power, $n$, of the product of two numbers, $a$ and $b$. The validity of the first is apparent when you simply write out the meaning of these symbols.

We give a demonstration for the case in which the exponents are natural numbers:

$$
\begin{array}{rlrl}
(a b)^{n} & =\underbrace{(a b) \cdot(a b) \cdot(a b) \cdot(a b) \cdots(a b)}_{n \text { factors }} & \triangleleft \text { defnition } \\
& =\underbrace{a \cdot a \cdot a \cdot a \cdots a}_{n \text { factors }} \underbrace{b \cdot b \cdot b \cdot b \cdots b}_{n \text { factors }} & \triangleleft \text { rearrange factors } \\
& =a^{n} b^{n} . & & \triangleleft \text { definition }
\end{array}
$$

This formula is valid for negative exponents as well. Let's look an example with a negative exponent, and 'prove' the validity of $(a b)^{n}=$ $a^{n} b^{n}$ by example:

$$
\begin{aligned}
(a b)^{-3} & =\frac{1}{(a b)^{3}} & & \triangleleft \text { defn of neg. exp. } \\
& =\frac{1}{a^{3} b^{3}} & & \triangleleft \text { Law \#2 } \\
& =a^{-3} b^{-3} & & \triangleleft \text { defn of neg. exp. }
\end{aligned}
$$

Section 2: Exponents \& Radicals

A Power of a Product: A power of the product of is the product of the powers of each. In symbols,

$$
(a b)^{n}=a^{n} b^{n}
$$

Using the formula $(a b)^{n}=a^{n} b^{n}$ from left-to-right is called expanding the expression. Here are some examples.

Illustration 7. Expand the following.
(a) $(x y)^{4}=x^{4} y^{4}$
(b) $(4 x)^{2}=4^{2} x^{2}=16 x^{2}$
(c) $4(x(y+1))^{23}=4 x^{23}(y+1)^{23}$
(d) $(a(3 b+5)(4 c-3))^{17}=a^{17}(3 b+5)^{17}(4 c-3)^{17}$
(e) $(4 x)^{-2}=4^{-2} x^{-2}=\frac{1}{4^{2}} \frac{1}{x^{2}}=\frac{1}{16 x^{2}}$
(f) $\frac{(s t)^{4}}{s^{3}}=\frac{s^{4} t^{4}}{s^{3}}=s t^{4}$ (by Cancellation Law).
(g) $a^{S^{6}} b^{7}(a b)^{-3}=a^{6} b^{7} a^{-3} b^{-3}=a^{6-3} b^{7-3}=a^{3} b^{4}$

Section 2: Exponents \& Radicals

The formula $(a b)^{n}=a^{n} b^{n}$ can be read from right-to-left as well. In this case, we are combining.

Illustration 8. Combine each of the following..
(a) $12 x^{12}(y+1)^{12}=12(x(y+1))^{12}$
(b) $a^{4} b^{4} c^{4}=(a b c)^{4}$
(c) $d^{m} p^{m} s^{m}=(d p s)^{m}$
(d) $\frac{1}{x^{9} y^{9}}=\frac{1}{(x y)^{9}}$
(e) $\frac{-3 x^{8} y^{8}}{(x y)^{10}}=\frac{-3(x y)^{8}}{(x y)^{10}}=\frac{-3}{(x y)^{2}}$

Now look of the 'quotient' version of the same law.

$$
\begin{aligned}
\left(\frac{a}{b}\right)^{n} & =\underbrace{\left(\frac{a}{b}\right) \cdot\left(\frac{a}{b}\right) \cdot\left(\frac{a}{b}\right) \cdot\left(\frac{a}{b}\right) \cdots\left(\frac{a}{b}\right)}_{n \text { factors }} & & \triangleleft \text { defnition } \\
& =\underbrace{\frac{a \cdot a \cdot a \cdot a \cdots a}{b \cdot b \cdot b \cdot b \cdots b}}_{n \text { factors }} & & \triangleleft \text { rearrange factors } \\
& =\frac{a^{n}}{b^{n}} & & \triangleleft \text { definition }
\end{aligned}
$$

Keep a mental visualization of this 'proof.' Thus,

A Power of a Quotient: A power of the quotient of is the quotient of the powers of each. In symbols,

$$
\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}
$$

Section 2: Exponents \& Radicals

Illustration 9. Now let's look at a few examples.
(a) $\left(\frac{x}{y}\right)^{4}=\frac{x^{4}}{y^{4}}$
(b) $\left(\frac{s}{s^{3}+1}\right)^{13}=\frac{s^{13}}{\left(s^{3}+1\right)^{13}}$
(c) $\left(\frac{x y}{x+y}\right)^{3}=\frac{(x y)^{3}}{(x+y)^{3}}=\frac{x^{3} y^{3}}{(x+y)^{3}}$
(d) $\left[\frac{x}{x^{2}+1}\right]^{-10}=\frac{x^{-10}}{\left(x^{2}+1\right)^{-10}}=\frac{\left(x^{2}+1\right)^{10}}{x^{10}}$.

Or, we can combine, instead of expanding.
(a) $\frac{x^{4}}{y^{4}}=\left(\frac{x}{y}\right)^{4}$
(b) $\frac{s^{13}}{\left(s^{3}+1\right)^{13}}=\left(\frac{s}{s^{3}+1}\right)^{13}$

Here's a few for you. Be neat. Use good notation. Be organized.

Section 2: Exponents \& Radicals

Exercise 2.5. Expand each of the following:
(a) $(a b)^{5}$
(b) $(5 x)^{3}$
(c) $(x(x+1))^{9}$
(d) $\left(\frac{w}{t}\right)^{6}$
(e) $\left[\frac{s(s+2)}{w}\right]^{7}$
(f) $\left[\frac{u+v}{u(4 u+v)}\right]^{-4}$

Exercise 2.6. Expand and combine, eliminating all negative exponents.

$$
\text { (a) } x^{4}(x y)^{-2} \text { (b) } \frac{(s t)^{-4}}{s^{3}} \text { (c) } x^{6}\left(\frac{x}{y}\right)^{-3}
$$

- How to Compute a Power of a Power

Students mix this property of exponents with the first Law of Exponents. Here we want to calculate a power of a power. The third Law of Exponents states that

$$
\left(a^{n}\right)^{m}=a^{n m}
$$

If you understand the origin behind these two laws, then you cannot get mixed up yourself. (We save that for someone else!) Take a gander at the 'proof.' From the definition, we have

$$
\begin{aligned}
\left(a^{n}\right)^{m} & =\underbrace{\underbrace{n} \cdot a^{n} \cdot a^{n} \cdot a^{n} \cdots a^{n}}_{m \text { factors }} \\
& =\overbrace{m \text { groups }}^{(a \cdot a \cdots a)} \cdot \overbrace{(a \cdot a \cdots a)}^{n \text { factors }} \cdots \overbrace{(a \cdot a \cdots a)}^{n \text { factors }} \\
& =\underbrace{a \cdot a \cdot a \cdot a \cdot a \cdots a}_{n m \text { factors }} \\
& =a^{n m}
\end{aligned}
$$

In the last step, we had $m$ groups of $n$ factors of $a$. The total number of factors of $a$ we have then is $n m$. This law is valid for negative exponents as well.

Illustration 10. Here are a sample of problems.
(a) (Skill Level 0) $\left(x^{5}\right)^{3}=x^{15},\left(x^{-5}\right)^{3}=x^{-15},\left(x^{5}\right)^{-3}=x^{-15}$, $\left(x^{-5}\right)^{-3}=x^{15}$. (All possible combinations of signs.) :-)
(b) Simplify $\left(x^{4} y^{6}\right)^{3}$.

Solution:

$$
\begin{aligned}
\left(x^{4} y^{6}\right)^{3} & =\left(x^{4}\right)^{3}\left(y^{6}\right)^{3} & & \triangleleft \text { Law \#2 } \\
& =x^{12} y^{18} & & \triangleleft \text { Law \#3 }
\end{aligned}
$$

Observe the use of the parentheses here. We want to protect ourselves against error. For example, if we raise $x^{4}$ to the $3^{\text {rd }}$ power, we denote that by $\left(x^{4}\right)^{3}$, which, in turn, simplifies to $x^{12}$.
(c) Simplify $\left(\frac{x^{5}}{y^{3}}\right)^{4}$.

Solution:

$$
\begin{align*}
\left(\frac{x^{5}}{y^{3}}\right)^{4} & =\frac{\left(x^{5}\right)^{4}}{\left(y^{3}\right)^{4}} & & \triangleleft \text { Law } \# 2 \\
& =\frac{x^{20}}{y^{12}} & & \triangleleft \text { Law } \# 3
\end{align*}
$$

Notice the use of the parentheses in, for example $\left(x^{5}\right)^{4}$. Without those parentheses, a mindless application of Law $\# 2$ would yield $x^{54}$ - this may be misinterpreted as the $54^{\text {th }}$ power, not as the $20^{\text {th }}$ power as it should.
(d) Simplify-eliminate negative exponents: $\left(x^{-4}(y-3)^{2}\right)^{-3}$. Solution:

$$
\begin{aligned}
\left(x^{-4}(y-3)^{2}\right)^{-3} & =x^{12}(y-3)^{-6} & & \triangleleft \text { Law } \# 3 \\
& =\frac{x^{12}}{(y-3)^{6}} & & \triangleleft \text { defn of } a^{-n}
\end{aligned}
$$

(e) Simplify $\left[\frac{x^{4}(s+1)^{2}}{5 w^{3}}\right]^{2}$.

Solution:

$$
\begin{aligned}
{\left[\frac{x^{4}(s+1)^{2}}{5 w^{3}}\right]^{2} } & =\frac{\left(x^{4}\right)^{2}\left((s+1)^{2}\right)^{2}}{5^{2}\left(w^{3}\right)^{2}} & & \triangleleft \text { Law \#2 } \\
& =\frac{x^{8}(s+1)^{4}}{25 w^{6}} & & \triangleleft \text { Law \#3 }
\end{aligned}
$$

Again not the judicious use of parentheses to protect oneself against possible error.
Quiz. Which of the following is a correct simplification to $\left(x^{4} y^{-4} z^{-3}\right)^{2}$
(a) $x^{6} y^{-2} z^{-1}$
(b) $x^{8} y^{-2} z^{-1}$
(c) $x^{8} y^{-8} z^{-6}$
(d) $x^{8} y^{-8} z^{-1}$

Before having you do a few problems, let's elevate the third LAW of Exponents to the status of a shadow box!

A Power of a Power: A power of a power of is the base raised to the product of the two exponents In symbols,

$$
\left(a^{n}\right)^{m}=a^{n m}
$$

Here is an exercise that uses all the Laws of Exponents. Use good techniques, be neat, and know how to justify each step.

ExERCISE 2.7. Simplify each of the following, removing any negative exponents as needed.

Section 2: Exponents \& Radicals
(a) $\left(a(a+1)^{2}(a+2)^{3}\right)^{4}$
(b) $\frac{(x y)^{6}}{\left(x^{3} y^{4}\right)^{2}}$
(c) $\left(x^{-1} y^{-1}\right)^{-3}$
(d) $\left(x^{4} y^{-2}\right)^{-3}$
(e) $\frac{2 s^{9} t^{3}}{(s t)^{5}}$
(f) $\frac{d^{-3} p^{3} s^{4}}{(d p s)^{-2}}$

### 2.3. Radicals

Life would be very tiresome if all we did in mathematics is to raise one number to an integer power. Let's strive forward and take up the concept of a root of a number.

Let $n \in \mathbb{N}$ be a natural number and $a \in \mathbb{R}$ be a real number. The $n^{\text {th }}$ root of the number $a$ is defined as follows.

Case 1: $n$ is an odd number. In this case the $n^{\text {th }}$ root of $a$ is defined to be that number $b \in \mathbb{R}$ such that $b^{n}=a$. In this case, write $b=\sqrt[n]{a}$. Thus,

$$
\begin{equation*}
b=\sqrt[n]{a} \quad \text { means } \quad b^{n}=a \tag{4}
\end{equation*}
$$

Case 2: $n$ is an even number. In this case, the $n^{\text {th }}$ root of $a$ is defined to be that number $b \geq 0$ such that $b^{n}=a$. In this case, write $b=\sqrt[n]{a}$. Thus,

$$
\begin{equation*}
b=\sqrt[n]{a} \quad \text { means } \quad b^{n}=a \tag{5}
\end{equation*}
$$

Note: If $a<0$, then there is no number $b \geq 0$ that satisfies (5). Even roots exist only for nonnegative numbers.

The expression $\sqrt[n]{a}$ is called a radical and $a$ is referred to as the radicand.

In the age of the electronic calculator, computing roots is trivial pursuit-actually you aren't computing roots you are approximating the roots. Therefore, we shall concentrate on the more algebraic or symbolic properties of radicals. But first, let's take a look at some simple examples and accompanying comments.

Illustration 11. There are a few numerical examples.
(a) (Case 1.) The odd root of any number exists: For example, $\sqrt[3]{8}=$ 2 , since $2^{3}=8$; and $\sqrt[3]{-8}=-2$, since $(-2)^{3}=-8$.
(b) (Case 2.) The even root of a negative number does not exist (as a real number). Thus, $\sqrt{-2}$, and $\sqrt[4]{-1.12345}$ do not exist as real numbers. (Note: they do exist as complex numbers).
(c) (Case 2.) The even root of a nonnegative number exists: For example, $\sqrt{4}=2$, since $2^{2}=4$; and $\sqrt[4]{81}=3$, since $3^{4}=81$.

Note that the even root is defined to be a nonnegative number. Here's an important point. By definition, the $\sqrt{4}$ is that number $b$ that satisfies three conditions:

$$
\text { (1) } b \in \mathbb{R} \quad \text { (2) } b \geq 0 \quad \text { (3) } b^{2}=4
$$

The student no doubt knows that there are two numbers $b$ that satisfy (1) and (3). The numbers 2 and -2 both have the property that their square is 4 ; however, only 2 satisfies all three conditions. It is 2 that we designate as $\sqrt{4}$; that is, $\sqrt{4}=2$. The number -2 is not referred to as the square root of 2 .

Sometimes we want to access both numbers. For example, suppose we want to find all solutions to the equation $b^{2}=33$. The correct
way of designing all solutions is $b=\sqrt{33}$ and $b=-\sqrt{33}$, or simply as $b= \pm \sqrt{33}$.

One last comment. Odd roots always exist and have the property that

$$
\begin{equation*}
\sqrt[n]{-a}=-\sqrt[n]{a} \quad \text { for } n \text { odd } \tag{6}
\end{equation*}
$$

This is a quite useful simplifying tool. The equalities $\sqrt[3]{-17}=-\sqrt[3]{17}$, and $\sqrt[5]{-(2 x+1)}=-\sqrt[5]{2 x+1}$ are two examples of the use of this property.

## - Properties of Radicals

When manipulating algebraic quantities, we often want to manipulate expressions involving radicals.

Let's begin with a quiz.
Quiz. Think about each of these questions before responding. When you click on the correct answer, you will jump to a discussion

1. Which of the following is a correct simplification of $\sqrt{a+b}$.
(a) $\sqrt{a}+\sqrt{b}$
(b) $a+b$
(c) $a^{2}+b^{2}$
(d) n.o.t.
2. Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$ a natural number. Is it (always) true that $\sqrt[n]{a^{n}}=a ?$
(a) True
(b) False

## End Quiz.

Properties of Radicals:
Let $a, b \in \mathbb{R}$ be real numbers, and $n, m \in \mathbb{N}$ be natural numbers, then

1. $\sqrt[n]{a^{n}}=a$, if $n$ is odd, and $\sqrt[n]{a^{n}}=|a|$, if $n$ is even.
2. $\sqrt[n]{a b}=\sqrt[n]{a} \sqrt[n]{b}$, if $n$ is odd, or if $n$ is even, provided $a \geq 0$ and $b \geq 0$.
3. $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, if $n$ is odd and $b \neq 0$, or if $n$ even, provided $a \geq 0$ and $b>0$.
4. $\sqrt[m]{\sqrt[n]{a}}=\sqrt[m n]{a}$, if $m$ and $n$ are both odd; otherwise, $a$ must be nonnegative $(a \geq 0)$.

Let's take a brief look at each of these in turn.

- The First Property of Radicals. The first stated property is usually no problem for students who already have experience working with radicals. When we are taking an odd root then

$$
\sqrt[n]{a^{n}}=a \quad n \text { odd }
$$

For example, $\sqrt[3]{27}=\sqrt[3]{3^{3}}=3, \sqrt[3]{x^{3}}=x, \sqrt[5]{(6 a-20)^{5}}=6 a-20$, and so on.

However, if we are taking an even root, then we must be a little more circumspect. The basic formula is

$$
\sqrt[n]{a^{n}}=|a| \quad n \text { even }
$$

Recall that an even root is always nonnegative; hence, the necessity for the absolute values.

We have no problems if $a$ is positive. The case where $a<0$ is the problem child. For example

$$
\sqrt{(-3)^{2}}=\sqrt{9}=3
$$

Notice that from this computation, $\sqrt{(-3)^{2}}=3 \neq-3$. Of course, 3 is related to -3 by $|-3|=3$. Thus we have shown in this example that $\sqrt{(-3)^{2}}=|-3|=3$.
Illustration 12. Consider the following phalanx of examples.
(a) $\sqrt{x^{2}}=|x|$, and $\sqrt[4]{(s+1)^{4}}=|s+1|$.
(b) $\sqrt{\cos ^{2}(x)}=|\cos (x)|$.
(c) If we have information related to the sign of the radicand, we can remove the absolute values.
a. $\sqrt{\left(x^{2}+1\right)^{2}}=\left|x^{2}+1\right|=x^{2}+1$, since $x^{2}+1>0$.
b. Suppose $s>3$, then $\sqrt{(s-3)^{2}}=|s-3|=s-3$, since $s-3>0$.
c. Suppose $t<1$, then $\sqrt{(t-1)^{2}}=|t-1|=-(t-1)=1-t$, since $t-1<0$.

Exercise 2.8. Simplify each of the following using the Properties of Radicals.
(a) $\sqrt[4]{(2 s+1)^{4}}$
(b) $\sqrt{\sin ^{2}\left(x^{2}\right)}$
(c) $\sqrt[7]{(x-10)^{7}}$
(d) $\sqrt[4]{(x-3)^{4}}$, given $x<3$.

Not all expressions are of the form $n^{\text {th }}$ roots of $n^{\text {th }}$ powers. An important variation is

$$
\sqrt[n]{a^{k n}}
$$

That is, $a$ is raised to an integer, $k$, multiple of $n$. The evaluation of this root depends on whether $n$ is odd or even and can be successfully carried out by the application of the Laws of Exponents.

Illustration 13. Consider the following.
(a) $\sqrt[3]{x^{6}}=\sqrt[3]{\left(x^{2}\right)^{3}}=x^{2}$, since $x^{6}=\left(x^{2}\right)^{3}$ by the Law of Exponents, Law 3.
(b) $\sqrt{x^{8}}=\sqrt{\left(x^{4}\right)^{2}}=\left|x^{4}\right|=x^{4}$. We inserted absolute values because we are extracting an even root. But since $x^{4} \geq 0$, we can, in turn, evaluate the absolute value to $x^{4}$.
(c) $\sqrt{y^{6}}=\sqrt{\left(y^{3}\right)^{2}}=\left|y^{3}\right|=|y|^{3}$. Here, $y^{3}$ may be positive or negative so we cannot remove the absolute values without information about the sign of $y$.

- The Second Property of Radicals. This property is very frequently used; care must be made not to abuse it. Property 2 states
that "a root of a product is the product of the roots, if $n$ is odd, or if $n$ is even, provided all factors are nonnegative."

The bad case is when $a<0$ and $b<0$ and we are taking an even root of $a b$; in this case,

$$
\sqrt[n]{a b} \neq \sqrt[n]{a} \sqrt[n]{b}
$$

To see this, put $n=2, a=-1$ and $b=-1$, to get

$$
\sqrt[2]{(-1)(-1)}=\sqrt[2]{1}=1 \neq \sqrt[2]{-1} \sqrt[2]{-1}
$$

We run into problems when we take an even root of a negative number. (A no, no.) This is rather obvious, but becomes less obvious when dealing with symbolic quantities. Whenever you are tempted, for example, to simplify an expression like $\sqrt{x^{2} y}$, you must be very careful not to write $x \sqrt{y}$.

Quiz. What is the simplification of $\sqrt{x^{2} y}$ ?
(a) $x \sqrt{y}$
(b) $|x| \sqrt{y}$
(c) I cannot be simplified

This second property is used, in combination with the first property of radicals to extract perfect roots.
Illustration 14. Extract perfect roots.
(a) $\sqrt{x(x+1)^{2}}=\sqrt{x} \sqrt{(x+1)^{2}}=\sqrt{x}|x+1|=|x+1| \sqrt{x}$.
(b) $\frac{x}{\sqrt[4]{x^{4} y}}=\frac{x}{\sqrt[4]{x^{4}}} \sqrt[4]{y}=\frac{x}{|x| \sqrt[4]{y}}$. (No additional simplification is possible unless more information is known about the sign of $x$.

1. If it is known that $x \geq 0$, then $|x|=x$ and so

$$
\frac{x}{|x| \sqrt[4]{y}}=\frac{x}{x \sqrt[4]{y}}=\frac{1}{\sqrt[4]{y}}
$$

2. If it is known that $x<0$, then $|x|=-x$ and so

$$
\frac{x}{|x| \sqrt[4]{y}}=\frac{x}{(-x) \sqrt[4]{y}}=\frac{1}{-\sqrt[4]{y}}=-\frac{1}{\sqrt[4]{y}}
$$

(c) $\sqrt{x^{4} y}=\sqrt{x^{4}} \sqrt{y}=\left|x^{2}\right| \sqrt{y}=x^{2} \sqrt{y}$.

Section 2: Exponents \& Radicals
(d) In this example, more detail is presented.

$$
\begin{aligned}
\sqrt{a^{4} b^{6}} & =\sqrt{a^{4}} \sqrt{b^{6}} & & \triangleleft \text { Prop. Rads. \#2 } \\
& =\sqrt{\left(a^{2}\right)^{2}} \sqrt{\left(b^{3}\right)^{2}} & & \triangleleft \text { Law Exp. \#3 } \\
& =a^{2}\left|b^{3}\right| & & \triangleleft \text { Prop. Rads. \#1 } \\
& =a^{2}|b|^{3} . & & \triangleleft \text { A Prop. Abs. Value \#2 }
\end{aligned}
$$

Illustration Notes: Illustration (d) is a good illustration how the various properties of exponents, of radicals, and of absolute values are utilized. When you become an expert, this simplification would be just a single step, not four steps as I have above.
Property \#2 can also be used to partial extraction. Let me illustrate what I mean.

Illustration 15. Study the following examples.
(a) $\sqrt{x^{3}}=\sqrt{x^{2} x}=\sqrt{x^{2}} \sqrt{x}=|x| \sqrt{x}=x \sqrt{x}$. In the last equality we were able to remove the absolute value symbols. Why?
(b) $\sqrt[3]{w^{10}}=\sqrt[3]{w^{9} w}=\sqrt[3]{w^{9}} \sqrt[3]{w}=w^{3} \sqrt[3]{w}$

- The Third Property of Radicals. This property is the same as \#2, but for quotients. Just two examples are sufficient.
Example 2.1. Simplify the following: $\sqrt{\frac{x^{8}}{y^{10}}}$.
Example 2.2. Simplify $\sqrt[3]{x^{-18}}$.
Exercise 2.9. Simplify each of the following utilizing the Properties of Radicals, the Laws of Exponents, and properties of Absolute Value.
(a) $\sqrt{25 x}$
(b) $\sqrt{w^{2} s^{6}}$
(c) $\sqrt{x^{5} y^{12}}$
(d) $\sqrt[5]{-w^{12}}$
(e) $\sqrt{\frac{a^{5}}{b^{6}}}$
(f) $\frac{\sqrt[3]{x^{6} y^{4}}}{\sqrt{x^{4} y^{6}}}$
- The Fourth Property of Radicals. This law describes what a root of a root is; it's . . . another root! Let's work it out. We want to understand the simplification of $\sqrt[m]{\sqrt[n]{a}}$. Let $b=\sqrt[m]{\sqrt[n]{a}}$. The characteristic property of $b$ is that $b^{m}=$ the radicand $=\sqrt[n]{a}$. Now if $b^{m}=\sqrt[n]{a}$, then by the definition, $\left(b^{m}\right)^{n}=$ the radicand $=a$. Thus,
$b^{m n}=a$. But this is the definition of $b$ being the $m n^{\text {th }}$ root of $a$; i.e., $b=\sqrt[m n]{a}$. We have shown that

$$
\sqrt[m]{\sqrt[n]{a}}=\sqrt[m n]{a}
$$

## Illustration 16.

(a) $\sqrt[3]{\sqrt[5]{x}}=\sqrt[15]{x}$.
(b) $\sqrt[4]{\sqrt{x^{10}}}=\sqrt[8]{x^{10}}=\sqrt[8]{x^{8} x^{2}}=\sqrt[8]{x^{8}} \sqrt[8]{x^{2}}=|x| \sqrt[8]{x^{2}}$.

- Question Argue that

$$
\sqrt[4]{\sqrt{x^{9}}}=\sqrt[8]{x^{9}}=x \sqrt[8]{x}
$$

is true. Compare this equation with (7) and explain why the absolute value is not needed.

### 2.4. Fractional Exponents

Let me finish off Lesson 2 by a (hopefully) brief discussion of fractional exponents. We begin by making a series of definitions.

Definition. Let $a$ be a number.

1. For $n$ a natural number, define $a^{1 / n}=\sqrt[n]{a}$, provided the root exists as discussed in the definition of radicals.
2. For $n$ a natural number, define $a^{-1 / n}=\frac{1}{a^{1 / n}}$.
3. If $p$ and $q$ are integers, define $a^{p / q}=\left(a^{p}\right)^{1 / q}$. (Of course, we are assuming $q \neq 0$.)

Illustration 17. Consider the following conversions from radical notation to exponential notation and from exponential notation to radical notation.
(a) $\sqrt[3]{x}=x^{1 / 3}$ and $\sqrt[6]{w^{2}+1}=\left(w^{2}+1\right)^{1 / 6}$.
(b) $\sqrt[5]{x^{-2}}=x^{-2 / 5}$ and $\sqrt[3]{-w^{-4}}=-\sqrt[3]{w^{-4}}=-w^{-4 / 3}=-\frac{1}{w^{4 / 3}}$.

Section 2: Exponents \& Radicals

$$
\begin{aligned}
& \text { (c) } s^{1 / 2}=\sqrt{s} \text { and } t^{-1 / 5}=\frac{1}{t^{1 / 5}}=\frac{1}{\sqrt[5]{t}} \\
& \text { (d) } \sqrt[3]{x^{2}}=x^{2 / 3}, \sqrt[2]{x^{5}}=x^{5 / 2} \text { and }(w+3)^{3 / 4}=\sqrt[4]{(w+3)^{3}} \\
& \text { (e) } x^{-2 / 7}=\frac{1}{x^{2 / 7}}=\frac{1}{\sqrt[7]{x^{2}}}
\end{aligned}
$$

Exercise 2.10. Convert each of the following to exponential notation using the above definitions.
(a) $\sqrt[4]{a}$
(b) $\sqrt[8]{s+t}$
(c) $\sqrt[4]{s^{3}}$
(d) $\sqrt[12]{x^{7}}$
(e) $\sqrt[3]{-x^{2}}$
(f) $\sqrt[4]{x^{-1}}$

Now let's convert from exponential to radical notation. Carefully work out the problems before you look at the solutions. Justify each step that you make in your solution.

Exercise 2.11. Convert to each to exponential notation.
(a) $4^{3 / 2}$
(b) $8^{-1 / 3}$
(c) $(-4)^{-3 / 2}$
(d) $(w+1)^{2 / 3}$
(e) $x^{-1 / 2}$
(f) $s^{-4 / 5}$

Here's an important point: The fractional exponent notation is entirely consistent with reduction of fractions to lowest terms. What do I mean by that?

Illustration 18. Reduction of Fractional Exponents.
(a) $x^{6 / 3}=x^{2}$ and $y^{-12 / 3}=y^{-4}$.
(b) $w^{4 / 12}=w^{1 / 3}$, for $w \geq 0$.
(c) $\sqrt[8]{x^{2}}=x^{2 / 8}=x^{1 / 4}=\sqrt[4]{x}$.

Quiz. Think about each of the following questions before responding.

1. The expression $\sqrt[12]{y^{3}}$ is equivalent to
(a) $\sqrt[9]{y}$
(b) $\sqrt[4]{y}$
(c) $\sqrt[3]{y}$
(d) n.o.t.
2. The expression $\sqrt[12]{y^{4}}$ is equivalent to
(a) $\sqrt[4]{|y|^{3}}$
(b) $\sqrt[3]{y^{2}}$
(c) $\sqrt[3]{|y|}$
(d) n.o.t.

## End Quiz.

The fractional exponent notation is entirely consistent with the Laws of Exponents. Let's restate this law for the record.

Section 2: Exponents \& Radicals

The Law of Exponents-Senior Grade:
Let $a$ and $b$ be numbers, and $r$ and $s$ be rational numbers, and assume further that $a^{r}$ and $a^{s}$ are defined. Then

1. $a^{r} a^{s}=a^{r+s}$
2. $(a b)^{r}=a^{r} a^{r}$ and $\left(\frac{a}{b}\right)^{r}=\frac{a^{r}}{a^{r}}$
3. $\left(a^{r}\right)^{s}=a^{r s}$

All the above equalities are conditional on the existence of each of the exponentials. For example, $(a b)^{r}=a^{r} b^{r}$ is true provided that each of the exponential $(a b)^{r}, a^{r}$, and $b^{r}$ exist. (Take a look at the second law above for the case $a=-1, b=-1$, and $r=1 / 2$-catastrophe!)

Exercise 2.12. Give examples of situations in which Law \#1 and Law \#3 are not valid.

In other words, all the techniques outlined and illustrated above applies to fractional exponents.

Illustration 19. Here are some examples that illustrate the Law of Exponents-Senior Grade.

1. Law 1: Different exponent, same base; just add exponents.
a. $x^{3 / 2} x^{5 / 2}=x^{3 / 2+5 / 2}=x^{8 / 2}=x^{4}$.
b. $w^{2 / 3} w^{-1 / 6}=w^{2 / 3-1 / 6}=w^{1 / 2}$
c. $\frac{x^{1 / 2} y^{3 / 2}}{4 x^{-1 / 2} y^{2}}=\frac{x^{1 / 2+1 / 2}}{4 y^{2-3 / 2}}=\frac{x}{4 y^{1 / 2}}=\frac{x}{4 \sqrt{y}}$.
d. $x^{7 / 2}=x^{3+1 / 2}=x^{3} x^{1 / 2}=x^{3} \sqrt{x}$.
2. Law 2: Same exponent, different base.
a. $(x y)^{2 / 3}=x^{2 / 3} y^{2 / 3}$.
b. $(s t)^{-3 / 4}=s^{-3 / 4} t^{-3 / 4}$. Here, we are taken even roots of $s$ and $t$; therefore, we must assume $s>0$ and $t>0$.
3. Law 3: Nested exponentials, just multiply exponents.
a. $4^{5 / 2}=\left(4^{1 / 2}\right)^{5}=(2)^{5}=32$. Whereas $4^{5 / 2}=\left(4^{5}\right)^{1 / 2}=$ $(1024)^{1 / 2}=32$. Naturally, the first computation was quite
a bit less complicated; the idea is to choose the order of calculation that is easiest.
b. $\left(x^{3}\right)^{2 / 3}=x^{2}$.
c. $\left(w^{-3 / 4}\right)^{-3}=w^{9 / 4}$.
d. Nested radicals are the same as nested exponents, let Exponent Law \#3 handle it.

$$
\sqrt[3]{\sqrt[4]{x}}=\left(x^{1 / 4}\right)^{1 / 3}=x^{1 / 12}=\sqrt[12]{x}
$$

EXERCISE 2.13. Let $n$ be an even positive integer. Simplify $\left(a^{n}\right)^{1 / n}$. Here are some routine practice exercises for you solve.

Exercise 2.14. Simplify each of the following using the Law of Exponents.
(a) $w^{-3 / 2} y^{2 / 3} w^{3} y^{1 / 2}$
(b) $\frac{x^{2 / 3}}{x^{3}}$
(c) $(4 x)^{-5 / 2}(4 x)^{2}$
(d) $(x y)^{3 / 2} x^{4}$
(e) $\frac{a^{1 / 2} b^{3 / 2}}{(a b)^{1 / 2}}$
(f) $\left(x^{3} y^{2 / 3}\right)^{4 / 3}$

Section 2: Exponents \& Radicals
This is the end of Lesson 2! If you have managed to work through it, you have my congratulations! $\mathfrak{P S}$

If you are not too exhausted, continue on to Lesson 3. See you there!

## Solutions to Exercises

2.1. Passing is $100 \%$ !
(a) $e^{-4}=\frac{1}{e^{4}}$.
(b) $\frac{1}{(x+y)^{-6}}=(x+y)^{6}$.
(c) $(s t)^{-(k+2)}=\frac{1}{(s t)^{k+2}}$.
(d) $\frac{w^{6}}{x^{4} y^{-7}}=\frac{w^{6} y^{7}}{x^{4}}$.
(e) $\frac{a^{7} b^{-9} c^{-1}}{d^{-4} e^{6}}=\frac{a^{7} d^{4}}{b^{9} c e^{6}}$.
(f) $\frac{(x+y)^{6}}{x^{-1} y^{-2}}=x y^{2}(x+y)^{6}$.

Solutions to Exercises (continued)
2.2. And the answers with any further comments are
(a) $x^{4} x^{4} y^{2}=x^{8} y^{2}$
(b) $(s+1)^{12}(s+1)(s+1)^{3}=(s+1)^{16}$
(c) $23 y^{3} w^{4} w^{8} y^{2}=23 y^{5} w^{12}$
(d) $4^{2} 4^{9}=4^{11}$
(e) $x^{4 m+3} x^{5 m-1}=x^{9 m+2}$
(f) $y^{3 j} y^{12 j+m}=y^{15 j+m}$
(g) $a^{6} a^{-3}=a^{3}$
(h) $(a b)^{7}(a b)^{4}(a b)^{-2}=(a b)^{7+4-2}=(a b)^{9}$
(i) $\frac{x^{5} y^{9} y^{-2}}{x^{2} y^{3}}=x^{5} y^{9} y^{-2} x^{-2} y^{-3}=x^{5-2} y^{9-2-3}=x^{3} y^{4}$

Add the exponents of all exponentials having the same base; bring factors in the denominator by changing the sign on the exponent, then add exponents as appropriate.

Solutions to Exercises (continued)
2.3. Solutions:
(a) $(-(w+1))^{13}=(-1)^{13}(w+1)^{13}=-(w+1)^{13}$.
(b) $(-3 w)^{4}=(-3)^{4} w^{4}=81 w^{4}$.
(c) $(-2 s)^{-3}=(-2)^{-3} s^{-3}=\frac{1}{(-2)^{3}} \frac{1}{s^{3}}=\frac{1}{-8} \frac{1}{s^{3}}=-\frac{1}{8 s^{3}}$.
(d) $(-u)^{2 n}=(-1)^{2 n} u^{2 n}=u^{2 n}$.
(e) $(-p)^{2 n-1}=(-1)^{2 n-1} p^{2 n-1}=-p^{2 n-1}$.

In the last problem, for any given integer $n, 2 n-1$ is an odd integer; hence, $(-1)^{2 n-1}=-1$

## Solutions to Exercises (continued)

2.4. Solutions:
(a) $\frac{4 x^{5} y^{6}}{x^{3} y^{9} z^{2}}=\frac{4 x^{2}}{y^{3} z^{2}}$.
(b) $\frac{w(u+v)^{9}}{w^{5}(u+v)^{3}}=\frac{(u+v)^{6}}{w^{4}}$.
(c) $\frac{x^{5} y^{5} z^{5}}{5 x^{4} y^{5} z^{12}}=\frac{x}{5 z^{7}}$.

Exercise Notes: In the case of solution (c), we automatically simplified $y^{5} / y^{5}$ to 1; i.e., $\frac{y^{5}}{y^{5}}=y^{0}=1$.

- Concentrate on using good techniques. Don't be sloppy. Don't revert to your old bad habits. Be precise in all things you do.

Exercise 2.4. -

Solutions to Exercises (continued)
2.5. Answer: Apply Law $\# 2$.
(a) $(a b)^{5}=a^{5} b^{5}$
(b) $(5 x)^{3}=5^{3} x^{3}=125 x^{3}$
(c) $(x(x+1))^{9}=x^{9}(x+1)^{9}$
(d) $\left(\frac{w}{t}\right)^{6}=\frac{w^{6}}{t^{6}}$
(e) $\left[\frac{s(s+2)}{w}\right]^{7}=\frac{(s(s+2))^{7}}{w^{7}}=\frac{s^{7}(s+2)^{7}}{w^{7}}$
(f) $\left[\frac{u+v}{u(4 u+v)}\right]^{-4}=\frac{(u+v)^{-4}}{u^{-4}(4 u+v)^{-4}}=\frac{u^{4}(4 u+v)^{4}}{(u+v)^{4}}$

In the last problem, the negative exponents were eliminated by move them to the numerator or denominator, as appropriate, and changing the signs of their exponents - standard techniques.
Did you use good notation? Were you neat? Were you organized?

Solutions to Exercises (continued)
2.6. Solutions: Apply combinations of Laws \#1 and \#2.
(a) $x^{4}(x y)^{-2}=x^{4} x^{-2} y^{-2}=x^{2} y^{-2}=\frac{x^{2}}{y^{2}}$
(b) $\frac{(s t)^{-4}}{s^{3}}=\frac{s^{-4} t^{-4}}{s^{3}}=\frac{1}{s^{4} t^{4} s^{3}}=\frac{1}{s^{7} t^{4}}$
(c) $x^{6}\left(\frac{x}{y}\right)^{-3}=x^{6} \frac{x^{-3}}{y^{-3}}=\frac{x^{6} x^{-3}}{y^{-3}}=x^{3} y^{3}$

Each of us develop our own pattern of thoughts; whatever pattern of thinking we develop, ultimately, it must be based on the simple application of these laws of exponents.

In these solutions, I've included a few more steps than is really used in practice.

Exercise 2.6.

Solutions to Exercises (continued)
2.7. Solutions:
(a) $\left(a(a+1)^{2}(a+2)^{3}\right)^{4}=a^{4}(a+1)^{8}(a+2)^{12}$.
(b) $\frac{(x y)^{6}}{\left(x^{3} y^{4}\right)^{2}}=\frac{x^{6} y^{6}}{x^{6} y^{8}}=\frac{1}{y^{2}}$
(c) $\left(x^{-1} y^{-1}\right)^{-3}=x^{3} y^{3}$
(d) $\left(x^{4} y^{-2}\right)^{-3}=x^{-12} y^{6}$
(e) $\frac{2 s^{9} t^{3}}{(s t)^{5}}=\frac{2 s^{9} t^{3}}{s^{5} t^{5}}=\frac{2 s^{4}}{t^{2}}$
(f) $\frac{d^{-3} p^{3} s^{4}}{(d p s)^{-2}}=\frac{d^{-3} p^{3} s^{4}}{d^{-2} p^{-2} s^{-2}}=\frac{p^{3} p^{2} s^{4} s^{2}}{d^{3} d^{-2}}=\frac{p^{5} s^{6}}{d}$

The last example requires a careful application of the Law of Exponents. Hope you were are careful as I was.

Did you work the problems out first? Were you neat? NOW is the time to develop good habits ... not during a test.

Solutions to Exercises (continued)
2.8. Solutions:
(a) $\sqrt[4]{(2 s+1)^{4}}=|2 s+1|$.
(b) $\sqrt{\sin ^{2}\left(x^{2}\right)}=\left|\sin \left(x^{2}\right)\right|$. Here the radicand is $\left(\sin \left(x^{2}\right)\right)^{2}$, hence, its square root is $\left|\sin \left(x^{2}\right)\right|$.
(c) $\sqrt[7]{(x-10)^{7}}=x-10$, since this is an odd root.
(d) Given $x<3, \sqrt[4]{(x-3)^{4}}=|x-3|=-(x-3)=3-x$, since $x-3<0$. (See definition of absolute value.)

Exercise 2.8.

Solutions to Exercises (continued)
2.9. Solutions:
(a) $\sqrt{25 x}=\sqrt{5^{2} x}=\sqrt{5^{2}} \sqrt{x}=5 \sqrt{x}$.
(b) $\sqrt{w^{2} s^{6}}=\sqrt{w^{2}} \sqrt{\left(s^{3}\right)^{2}}=|w|\left|s^{3}\right|=|w||s|^{3}$.
(c) $\sqrt{x^{5} y^{12}}=\sqrt{x^{5}} \sqrt{y^{12}}=\sqrt{x^{4} x}\left|y^{6}\right|=\sqrt{x^{4}} \sqrt{x} y^{6}=x^{2} \sqrt{x} y^{6}$. Here we have used the fact that $y^{6} \geq 0$, hence, $\left|y^{6}\right|=y^{6}$; similarly, $\sqrt{x^{4}}=\left|x^{2}\right|=x^{2}$, since $x^{2} \geq 0$.

(e) $\sqrt{\frac{a^{5}}{b^{6}}}=\frac{\sqrt{a^{5}}}{\sqrt{b^{6}}}=\frac{\sqrt{a^{4}} \sqrt{a}}{|b|^{3}}=\frac{a^{2} \sqrt{a}}{|b|^{3}}$.
(f) $\frac{\sqrt[3]{x^{6} y^{4}}}{\sqrt{x^{4} y^{6}}}=\frac{\sqrt[3]{x^{6}} \sqrt[3]{y^{4}}}{\sqrt{x^{4}} \sqrt{y^{6}}}=\frac{x^{2} \sqrt[3]{y^{3} y}}{x^{2} \sqrt{\left(y^{3}\right)^{2}}}=\frac{x^{2} y \sqrt[3]{y}}{x^{2}\left|y^{3}\right|}=\frac{y \sqrt[3]{y}}{|y|^{3}}$.

In problem (f), no further simplification is possible without knowledge of the sign of $y$.

Solutions to Exercises (continued)
2.10. Solutions:
(a) $\sqrt[4]{a}=a^{1 / 4}$.
(b) $\sqrt[8]{s+t}=(s+t)^{1 / 8}$.
(c) $\sqrt[4]{s^{3}}=s^{3 / 4}$.
(d) $\sqrt[12]{x^{7}}=x^{7 / 12}$.
(e) $\sqrt[3]{-x^{2}}=-\sqrt[3]{x^{2}}=-x^{2 / 3}$.
(f) $\sqrt[4]{x^{-1}}=x^{-1 / 4}$.

Conversion to exponential must be automatic.

Solutions to Exercises (continued)
2.11. Solutions:
(a) $4^{3 / 2}=\sqrt{4^{3}}=\sqrt{64}=8$.
(b) $8^{-1 / 3}=\frac{1}{8^{1 / 3}}=\frac{1}{\sqrt[3]{8}}=\frac{1}{2}$.
(c) $(-4)^{-3 / 2}=\sqrt{(-4)^{3}}=\sqrt{-64}$. Oops! An even root of a negative number! Therefore, $(-4)^{-3 / 2}$ does not evaluate to a real number;
It evaluates to a complex number. Did you panic?
(d) $(w+1)^{2 / 3}=\sqrt[3]{(w+1)^{2}}$.
(e) $x^{-1 / 2}=\frac{1}{x^{1 / 2}}=\frac{1}{\sqrt{x}}$.
(f) $s^{-4 / 5}=\frac{1}{s^{4 / 5}}=\frac{1}{\sqrt[5]{s^{4}}}$.

Solutions to Exercises (continued)
2.12. Law $\# 1$. Consider the case $a=-1, r=1 / 2$ and $s=-1 / 2$. In this case, $a^{r+s}=(-1)^{1 / 2-1 / 2}=(-1)^{0}=1$, yet $a^{r}=(-1)^{1 / 2}$ and $a^{s}=(-1)^{-1 / 2}$ are not defined as a real number.

Law \#3. Take $a=-1, r=1 / 2$ and $s=2$, then

$$
\left(a^{1 / 2}\right)^{2} \neq a^{(1 / 2)(2)}
$$

The right-hand side is $a^{1}=a=-1$, whereas the left-hand side has in it $a^{1 / 2}=(-1)^{1 / 2}$, which is not a real number.

In each of the two examples, one of the expressions is not a real number ... what's where we ran into trouble. The Laws are valid whenever all exponentials are defined.

Solutions to Exercises (continued)
2.13. This is another exception to Law $\# 3$.

$$
\left(a^{n}\right)^{1 / n}=\sqrt[n]{a^{n}}=|a|
$$

by the (1) of the property of radicals. Thus, if $a<0$, then

$$
\left(a^{n}\right)^{1 / n}=|a| \neq a .
$$

Doesn't this contradict the third law of exponents? The premise of the laws of the exponents is that $a^{r}$ and $a^{s}$ are both defined. In this situation, for $a<0, a^{1 / n}$ is not defined since we are taking an even root of a negative number; therefore, the Laws of the Exponents does not guarantee the validity of Law \#3.

Solutions to Exercises (continued)
2.14. Solutions:
(a) $w^{-3 / 2} y^{2 / 3} w^{3} y^{1 / 2}=w^{(-3 / 2+3)} y^{(2 / 3+1 / 2)}=w^{3 / 2} y^{7 / 6}$.
(b) $\frac{x^{2 / 3}}{x^{3}}=\frac{1}{x^{(3-2 / 3)}}=\frac{1}{x^{7 / 3}}$.
(c) $(4 x)^{-5 / 2}(4 x)^{2}=(4 x)^{(-5 / 2+2)}=(4 x)^{-1 / 2}=\frac{1}{(4 x)^{1 / 2}}=\frac{1}{2 \sqrt{x}}$.
(d) $(x y)^{3 / 2} x^{4}=x^{3 / 2} y^{3 / 2} x^{4}=x^{(3 / 2+4)} y^{3 / 2}=x^{11 / 2} y^{3 / 2}$.
(e) $\frac{a^{1 / 2} b^{3 / 2}}{(a b)^{1 / 2}}=\frac{a^{1 / 2} b^{3 / 2}}{a^{1 / 2} b^{1 / 2}}=b^{(3 / 2-1 / 2)}=b$.
(f) $\left(x^{3} y^{2 / 3}\right)^{4 / 3}=\left(x^{3}\right)^{4 / 3}\left(y^{2 / 3}\right)^{4 / 3}=x^{4} y^{8 / 9}$.

## Solutions to Examples

2.1. Solution:

$$
\begin{align*}
\sqrt{\frac{x^{8}}{y^{10}}} & =\frac{\sqrt{x^{8}}}{\sqrt{y^{10}}} & & \triangleleft \text { Prop. Rads. \#3 } \\
& =\frac{x^{4}}{\left|y^{5}\right|} & & \triangleleft \text { Prop. Rads. \#1 }  \tag{S-1}\\
& =\frac{x^{4}}{|y|^{5}} & & \triangleleft \text { A Prop. Abs. Value \#2 }
\end{align*}
$$

Example Notes: In equation (S-1), no absolute values were needed for the $x^{4}$ since $x^{4} \geq 0$; however, it is possible for $y^{5}$ to be negative, hence the absolute values were needed.

Solutions to Examples (continued)
2.2. Solution:

$$
\begin{aligned}
\sqrt[3]{x^{-18}} & =\sqrt[3]{\frac{1}{x^{18}}} & & \triangleleft \text { defn of } a^{-n} \\
& =\frac{1}{\sqrt[3]{x^{18}}} & & \triangleleft \text { Prop. Rads. } \# 3 \\
& =\frac{1}{\sqrt[3]{\left(x^{6}\right)^{3}}} & & \triangleleft \text { Law Exp. } \# 3 \\
& =\frac{1}{x^{6}} & & \triangleleft \text { Prop. Rads. } \# 1
\end{aligned}
$$

## Important Points

Important Points (continued)

That's Right! The correct answer is (b).
The square of $-3=(-3)^{2}=(-3)(-3)=9$
First choice or second? If you chose (a) as your first choice, you have a weakness in this area. Some students erroneously write $-3^{2}$ when what they really mean is $(-3)^{2}$. A mathematician would interpret the expression $-3^{2}$ as $-(3)(3)=-9$; that is the correct interpretation of the notation $-3^{2}$ is the negative of the number 3 squared.

You must be careful about writing $-3^{2}$ when, in reality, you mean $(-3)^{2}$. You and the one grading your paper might have a difference in opinion about the meaning of the symbol-guess who wins the argument.

Therefore, when raising a number that has a negative sign, always enclose the number, including negative sign, with parentheses. Thus, cube of the number -2 is $(-2)^{3}=(-2)(-2)(-2)=-8$.

Important Points (continued)
Similar comments can be made about symbolic numbers: The cube of the number $-x$ is $(-x)^{3}$ not $-x^{3}$. More on this later.

Important Point •

Important Points (continued)

Good Choice! The correct response is 'n.o.t.', which means 'none of these.'

A common error students make when (trying) to manipulate radicals is to essentially write (on a test paper, for example)

$$
\text { NOT TRUE }!\Longrightarrow \sqrt{a+b}=\sqrt{a}+\sqrt{b} \Longleftarrow \text { NOT TRUE! }
$$

But this is an algebraic blunder! Don't do that! The root of a sum of two quantities in not equal to the sum of the roots. A simple example will illustrate

Symbolic quantities such as $a$ and $b$ represent numbers. Equations you write involving symbolic quantities must be true when the symbols are replaced with numbers. To see that the root of a sum is not equal to the sum of the roots, just give $a$ and $b$ appropriate values.

Take $a=9$ and $b=16$. Thus,

$$
\sqrt{9+16}=\sqrt{25}=5 \neq 7=3+4=\sqrt{9}+\sqrt{16}
$$

Important Points (continued)
Look at the extreme left and right sides of this nonequation. What do you see?

$$
\sqrt{9+16} \neq \sqrt{9}+\sqrt{16}
$$

That is,

$$
\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}
$$

Don't make this mistake ever again!

Correct Again! It is not always true that $\sqrt[n]{a^{n}}=a$. Sometimes the radical $\sqrt[n]{a^{n}}$ equals $a$ and sometimes it doesn't equal $a$. Here are some examples.

Example 1. Situation where $\sqrt[n]{a^{n}}=a$ is false. Think of $a=-1$ and $n=2$, then

$$
\sqrt[2]{a^{n}}=\sqrt[2]{(-1)^{2}}=\sqrt[2]{1}=1 \neq-1=a
$$

Example 2. Situation where $\sqrt[n]{a^{n}}=a$ is true. Think of $a=1$ and $n=2$, then

$$
\sqrt[2]{a^{n}}=\sqrt[2]{1^{2}}=\sqrt[2]{1}=1=a
$$

Sometimes it true, sometimes its false. When trying to simplifying a symbolic expression like $\sqrt[n]{a^{n}}$, it turns out that we need to know the sign of the number $a$.

Continue now the discussion following this quiz for a definitive explanation of how to simplify radicals like $\sqrt[n]{a^{n}}$.

Way to go, mate! Because we are taking the square root of $x^{2} y$, it is an implicit assumption that $x^{2} y \geq 0$. Since $x^{2} \geq 0$ regardless of the value of $x$, we deduce $y \geq 0$.

Now by Law \#2 of the Properties of Radicals we have

$$
\begin{aligned}
\sqrt{x^{2} y} & =\sqrt{x^{2}} \sqrt{y} \\
& =|x| \sqrt{y} \quad \triangleleft \text { Law \#1 }
\end{aligned}
$$

Important Points (continued)
The original expression was $\sqrt{x^{3}}$, from which we deduce that $x^{3} \geq 0$; hence, $x \geq 0$. Thus, $|x|=x$ since $x \geq 0$. 'Nuff Said!

Important Points (continued)
Solution:

$$
\begin{aligned}
\sqrt[4]{\sqrt{x^{9}}} & =\sqrt[8]{x^{9}} \\
& =\sqrt[8]{x^{8} x} \\
& =\sqrt[8]{x^{8}} \sqrt[8]{x} \\
& =|x| \sqrt[8]{x} \\
& =x \sqrt[8]{x}
\end{aligned}
$$

The last step needs some comment. Because we started with the expression $\sqrt[8]{x^{9}}$, we conclude that $x^{9} \geq 0$. But $x^{9} \geq 0$ implies that $x \geq 0$, because we are dealing with an odd power of $x$.

Finally, $x \geq 0$ implies $|x|=x$. What was simple!
Awareness of the signs of the quantities is often essential to a successful simplification.

Way to go! Convert to exponential notation, reduce fractions, then return to radical notation.

$$
\sqrt[12]{y^{4}}=y^{4 / 12}=y^{1 / 3}=\sqrt[3]{y}
$$

